Cohomological finite generation for the group scheme SL_2.

Wilberd van der Kallen

Dedicated to the memory of T. A. Springer

Abstract

Let G be the group scheme SL_2 defined over a noetherian ring **k**. If G acts on a finitely generated commutative **k**-algebra A, then $H^*(G, A)$ is a finitely generated **k**-algebra.

1 Introduction

Let **k** be a noetherian ring. Consider a flat linear algebraic group scheme G defined over **k**. Recall that G has the cohomological finite generation property (CFG) if the following holds: Let A be a finitely generated commutative **k**-algebra on which G acts rationally by **k**-algebra automorphisms. (So G acts from the right on $\mathrm{Spec}(A)$.) Then the cohomology ring $H^*(G, A)$ is finitely generated as a **k**-algebra. Here, as in [3, I.4], we use the cohomology introduced by Hochschild, also known as 'rational cohomology'.

This note is part of the project of studying (CFG) for reductive G. Recall that the breakthrough of Touzé [4] settled the case when **k** is a field [6]. And [7, Theorem 10.1] extended this to the case that **k** contains a field. In this paper we show that in the case $G = SL_2$ one can dispense with the condition that **k** contains a field. According to the last item of [7, Theorem 10.5] it suffices to show that $H^*(G, A/pA)$ is a noetherian module over $H^*(G, A)$ whenever p is a prime number. We fix p. To prove the noetherian property we employ universal cohomology classes as in earlier work. More specifically, we lift the cohomology classes $c_r[a]^{(j)}$ of [5, 4.6] to classes in cohomology of SL_2 over the integers with flat coefficient module $\Gamma^m \Gamma^{p^{r+j}}(\mathfrak{gl}_2)$. We get the lifts with explicit formulas that do not seem to generalize to SL_n with $n > 2$.

Once we have the lifts of the cohomology classes we can lift enough of the mod p constructions to conclude that $H^*(G, A)$ hits much of $H^*(G, A/pA)$. As $H^*(G, A/pA)$ itself is a finitely generated **k**-algebra this will then imply that $H^*(G, A/pA)$ is a noetherian module over $H^*(G, A)$.

For simplicity of reference we use [7]. As we are working with SL_2 that amounts to serious overkill. For instance, the work of Touzé is not needed for SL_2. Further the 'functorial resolution of the ideal of the diagonal in a product of Grassmannians' now just means that the ideal sheaf of the diagonal divisor in a product of two projective lines is the familiar line bundle $\mathcal{O}(-1) \boxtimes \mathcal{O}(-1)$. And Kempf vanishing for SL_2 is immediate from the computation of the cohomology of line bundles on $\mathbb{P}^1$.

2 Rank one

We take $G = SL_2$ as group scheme over the noetherian ring **k**. Initially **k** is just $\mathbb{Z}$. Let T be the diagonal torus and B the Borel subgroup of lower triangular matrices. Its root α is the negative root.

2.1 Cocycles for the additive group.

We have fixed a prime p. Define $\Phi(X, Y) \in \mathbb{Z}[X, Y]$ by

$$(X + Y)^p = X^p + Y^p + p\Phi(X, Y).$$

By induction one gets for $r \geq 1$

$$(X + Y)^{p^r} \equiv X^{p^r} + Y^{p^r} + p\Phi(X^{p^{r-1}}, Y^{p^{r-1}}) \bmod p^2.$$

Put

$$c_r^{\mathbb{Z}}(X, Y) = \frac{(X + Y)^{p^r} - X^{p^r} - Y^{p^r}}{p} \in \mathbb{Z}[X, Y].$$

We think of $c_r^{\mathbb{Z}}$ as a 2-cochain in the Hochschild complex $C^\bullet(\mathbb{G}_a, \mathbb{Z})$ as treated in [3, I 4.14, I 4.20]. Then $c_r^{\mathbb{Z}}$ is a 2-cocycle because $pc_r^{\mathbb{Z}}$ is a coboundary. One has

$$c_r^{\mathbb{Z}}(X, Y) \equiv \Phi(X^{p^{r-1}}, Y^{p^{r-1}}) \bmod p.$$

Taking cup products one finds a $2m$-cocycle $c_r^{\mathbb{Z}}(X, Y)^{\cup m}$ representing a class in $H^{2m}(\mathbb{G}_a, \mathbb{Z})$. The cocycle $c_r^{\mathbb{Z}}$ serves as lift of the $(r-1)$-st Frobenius twist

of the Witt vector class that was our starting point in [5, §4]. We can now follow [5, §4], lifting all relevant mod p constructions to the integers. That will do the trick.

2.2 Universal classes

Our next task is to construct a universal class $c_r[m]^{(j)}$ in $H^{2mp^{r-1}}(G, \Gamma^m \Gamma^{p^{r+j}}(\mathfrak{gl}_2))$.

Let $r \geq 1$, $j \geq 0$, $m \geq 1$. Let α be the negative root, and let $x_\alpha : \mathbb{G}_a \to SL_2$ be its root homomorphism, with image U_α. For a $\mathbb{Z}$-module V its m-th module of divided powers is written $\Gamma^m V$ and its dual $\mathrm{Hom}_\mathbb{Z}(V, \mathbb{Z})$ is written $V^\#$.

Consider the representation $\Gamma^{mp^{r+j}}(\mathfrak{gl}_2)$ of G with its restriction $x_\alpha^* \Gamma^{mp^{r+j}}(\mathfrak{gl}_2)$ to $\mathbb{G}_a$. Its lowest weight is $mp^{r+j}\alpha$. Say e_α is the elementary matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ that spans the α weight space of $\mathfrak{gl}_2$, and $e_\alpha^{[mp^{r+j}]}$ denotes its divided power in $\Gamma^{mp^{r+j}}(\mathfrak{gl}_2)$. Then $c_{j+1}^\mathbb{Z}(X,Y)^{\cup mp^{r-1}} e_\alpha^{[mp^{r+j}]}$ represents a class in $H^{2mp^{r-1}}(\mathbb{G}_a, x_\alpha^* \Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$ and the corresponding element of $H^{2mp^{r-1}}(U_\alpha, \Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$ is T-invariant. So we get a class in $H^{2mp^{r-1}}(B, \Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$ and by Kempf vanishing ([3, II B.3] with $\lambda = 0$) a class in $H^{2mp^{r-1}}(G, \Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$. Recall that one obtains a natural map from $\Gamma^{p^{r+j}}(\mathfrak{gl}_2 \bmod p)$ to the $(r+j)$-th Frobenius twist $(\mathfrak{gl}_2 \bmod p)^{(r+j)}$ by dualizing the map from $(\mathfrak{gl}_2^\# \bmod p)$ to $S^{p^{r+j}}(\mathfrak{gl}_2^\# \bmod p)$ that raises a vector $v \in (\mathfrak{gl}_2^\# \bmod p)$ to its p^{r+j}-th power. So $\Gamma^{mp^{r+j}}(\mathfrak{gl}_2)$ maps naturally to $\Gamma^m((\mathfrak{gl}_2 \bmod p)^{(r+j)})$ by way of $\Gamma^m \Gamma^{p^{r+j}}(\mathfrak{gl}_2)$. Applying this to our class in $H^{2mp^{r-1}}(G, \Gamma^{mp^{r+j}}(\mathfrak{gl}_2))$ we hit a class in $H^{2mp^{r-1}}(G, \Gamma^m((\mathfrak{gl}_2 \bmod p)^{(r+j)}))$, which is where $c_r[m]^{(j)}$ of [5, 4.6] lives. On the root subgroup $U_\alpha \bmod p$ it is given by the cocycle $\Phi(X^{p^j}, Y^{p^j})^{\cup mp^{r-1}} e_\alpha^{(r+j)[m]} \bmod p$, where $e_\alpha^{(r+j)[m]} \bmod p$ is our notation for the obvious basis vector of the lowest weight space of $\Gamma^m((\mathfrak{gl}_2 \bmod p)^{(r+j)})$. This cocycle is the same as the one used in [5, 4.6] to construct $c_r[m]^{(j)}$. But then their cohomology classes agree on B and G also. So we have lifted the $c_r[m]^{(j)}$ of [5, 4.6] to a cohomology group with a coefficient module $\Gamma^m \Gamma^{p^{r+j}}(\mathfrak{gl}_2)$ that is flat over the integers.

Notation 2.3 Simply write $c_r[m]^{(j)}$ for the lift in $H^{2mp^{r-1}}(G, \Gamma^m \Gamma^{p^{r+j}}(\mathfrak{gl}_2))$.

2.4 Pairings

In [5, 4.7] we used the pairing between the modules $\Gamma^m(\mathfrak{gl}_2 \bmod p)^{(r)}$ and $S^m(\mathfrak{gl}_2^\# \bmod p)^{(r)}$. We want to lift it to a pairing between representations $\Gamma^m(X_r)$ and $S^m(Y_r)$ of G over $\mathbb{Z}$. We take $X = X_r = \Gamma^{p^r}(\mathfrak{gl}_2)$ and define $K = \ker(X \to (\mathfrak{gl}_2 \bmod p)^{(r)})$.

Put $Y = Y_r = \ker(\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(K, \mathbb{Z}/p\mathbb{Z}))$. Then $Y \to \mathrm{Hom}_{\mathbb{Z}}((X/K), \mathbb{Z}/p\mathbb{Z})$ is surjective because X is a free $\mathbb{Z}$-module. Notice that $\mathrm{Hom}_{\mathbb{Z}}((X/K), \mathbb{Z}/p\mathbb{Z})$ is just $(\mathfrak{gl}_2^\# \bmod p)^{(r)}$. Thus Y_r is flat and maps onto $(\mathfrak{gl}_2^\# \bmod p)^{(r)}$.

We have a commutative diagram

$$
\begin{array}{ccc}
\Gamma^m X \otimes S^m Y & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \\
\Gamma^m((\mathfrak{gl}_2 \bmod p)^{(r)}) \otimes S^m(\mathfrak{gl}_2^\# \bmod p)^{(r)} & \longrightarrow & \mathbb{Z}/p\mathbb{Z}
\end{array}
$$

and the left vertical arrow is surjective. So we have found our lift of the pairing from [5, 4.7].

Remark 2.5 Notice that we do not use the precise shape of X here. What matters is that X is free over $\mathbb{Z}$, with a surjection of G modules $X \to (\mathfrak{gl}_2 \bmod p)^{(r)}$, and that, for $1 \leq i \leq r$, we have an element in $H^{2mp^{i-1}}(G, \Gamma^m X)$, suggestively denoted $c_i[m]^{(r-i)}$, that is mapped to the $c_i[m]^{(r-i)}$ of [5] under the map induced by $X \to (\mathfrak{gl}_2 \bmod p)^{(r)}$.

2.6 Noetherian base ring

From now on let $\mathbf{k}$ be an arbitrary commutative noetherian ring. By base change to $\mathbf{k}$ we get a group scheme over $\mathbf{k}$ that we write again as $G = SL_2$. We simply write X_r for $X_r \otimes_{\mathbb{Z}} \mathbf{k}$ and we write Y_r for $Y_r \otimes_{\mathbb{Z}} \mathbf{k}$. We keep suppressing the base ring $\mathbf{k}$ in most notations, so that $X_r = \Gamma^{p^r}(\mathfrak{gl}_2)$, with classes $c_i[m]^{(r-i)}$ in $H^{2mp^{i-1}}(G, \Gamma^m X_r)$. The commutative diagram above becomes after base change

$$
\begin{array}{ccc}
\Gamma^m X_r \otimes S^m Y_r & \longrightarrow & \mathbf{k} \\
\downarrow & & \downarrow \\
\Gamma^m((\mathfrak{gl}_2 \bmod p)^{(r)}) \otimes S^m((\mathfrak{gl}_2^\# \bmod p)^{(r)}) & \longrightarrow & \mathbf{k} \bmod p
\end{array}
$$

Lemma 2.7 *If V is a representations of G and $v \in V$, then the subrepresentation generated by v exists and is finitely generated as a k-module.*

Proof As $\mathbf{k}[G]$ is a free $\mathbf{k}$-module, this follows from [SGA3, Exposé VI, Lemme 11.8]. □

2.8 Cup products from pairings

Let U, V, W, Z be G-modules, and $\phi : U \otimes V \to Z$ a G-module map. We call ϕ a pairing. Computing with Hochschild complexes one gets cup products $H^i(G, U) \otimes H^j(G, V \otimes W) \to H^{i+j}(G, Z \otimes W)$ induced by ϕ. Note that we are not assuming that the modules are flat over $\mathbf{k}$. We think of the Hochschild complex for computing $H^i(G, M)$ as $(C^*(G, \mathbf{k}[G]) \otimes M)^G$, where $C^*(G, \mathbf{k}[G])$ has a differential graded algebra structure as described in [6, section 6.3].

2.9 Hitting invariant classes

Definition 2.10 Recall that we call a homomorphism of $\mathbf{k}$-algebras $f : A \to B$ *noetherian* if f makes B into a noetherian left A-module. It is called *power surjective* [2, Definition 2.1] if for every $b \in B$ there is an $n \geq 1$ so that the power b^n is in the image of f.

See [6, Section 6.2] for some relevant properties of noetherian maps in cohomology. We are now going to look for noetherian maps. We keep the prime p fixed. Let $r \geq 1$. Let $\bar{G}$ denote G base changed to $(\mathbf{k} \bmod p)$, and let $\bar{G}_r$ denote its r-th Frobenius kernel. (Here we use that $\bar{G}$ is defined over $\mathbb{Z}/p\mathbb{Z}$.) We use bars to indicate structures having $(\mathbf{k} \bmod p)$ as base ring. Let $\bar{C}$ be a finitely generated commutative $(\mathbf{k} \bmod p)$-algebra with $\bar{G}$ action on which $\bar{G}_r$ acts trivially. By [2, Remark 52] we may view $\bar{C}$ also as an algebra with G action. Let $\mathcal{C}$ be a finitely generated commutative $\mathbf{k}$-algebra with G action and let $\pi : \mathcal{C} \to \bar{C}$ be a power surjective equivariant homomorphism.

Theorem 2.11 $H^{\mathrm{even}}(G, \mathcal{C}) \to H^0(G, H^*(\bar{G}_r, \bar{C}))$ *is noetherian.*

Proof By [1, Thm 1.5, Remark 1.5.1] $H^*(\bar{G}_r, \bar{C})$ is a noetherian module over the finitely generated graded algebra

$$\bar{R} = \bigotimes_{a=1}^{r} S^*((\bar{\mathfrak{gl}}_2^{(r)})^{\#}(2p^{a-1})) \otimes \bar{C}.$$

Here $(\bar{\mathfrak{gl}}_2^{(r)})\#(2p^{a-1})$ means that one places a copy of $(\bar{\mathfrak{gl}}_2^{(r)})\#$ in degree $2p^{a-1}$. It is easy to see that the obvious map from $\mathcal{R} = \bigotimes_{a=1}^r S^*(Y_r(2p^{a-1})) \otimes \mathcal{C}$ to $\bar{R}$ is noetherian. So by invariant theory [2, Thm. 9], $H^0(G, H^*(\bar{G}_r, \bar{C}))$ is a noetherian module over the finitely generated algebra $H^0(G, \mathcal{R})$. By [6, Remark 6.7] it now suffices to factor the map $H^0(G, \mathcal{R}) \to H^0(G, H^*(\bar{G}_r, \bar{C}))$ as a set map through $H^{\text{even}}(G, \mathcal{C}) \to H^0(G, H^*(\bar{G}_r, \bar{C}))$.

On a summand

$$H^0(G, \bigotimes_{a=1}^r S^{i_a}(Y_r(2p^{a-1})) \otimes \mathcal{C})$$

of $H^0(G, \mathcal{R})$ we simply take cup product with the (lifted) $c_a[i_a]^{(r-a)}$ according to the pairing of $S^{i_a}(Y_r)$ with $\Gamma^{i_a}(X_r) = \Gamma^{i_a}\Gamma^{p^r}(\mathfrak{gl}_2)$. In the proof of [5, Cor. 4.8] one has a similar description of the map to $H^*(\bar{G}_r, \bar{C})$ on the summand

$$H^0(G, \bigotimes_{a=1}^r S^{i_a}((\bar{\mathfrak{gl}}_2^{(r)})\#(2p^{a-1})) \otimes \bar{C})$$

of $H^0(G, R)$. The required factoring as a set map thus follows from the compatibility of the pairings and the fact that the lifted $c_a[i_a]^{(r-a)}$ are lifts of their mod p namesakes. □

Recall that G is the group scheme SL_2 over the noetherian base ring $\mathbf{k}$. Now let A be a finitely generated commutative $\mathbf{k}$-algebra with G action.

Theorem 2.12 (CFG in rank one) $H^*(G, A)$ *is a finitely generated algebra.*

Proof Recall that A comes with an increasing filtration $A_{\leq 0} \subseteq A_{\leq 1} \cdots$ where $A_{\leq i}$ denotes the largest G-submodule all whose weights λ satisfy $\text{ht}\,\lambda = \sum_{\beta > 0}\langle \beta, \beta^\vee \rangle \leq i$. (Actually there is now only one positive root, so that the sum has just one term.) The associated graded algebra is the Grosshans graded ring $\text{gr}\,A$. Let $\mathcal{A}$ be the Rees ring of the filtration. So $\mathcal{A}$ is the subring of the polynomial ring $A[t]$ generated by the subsets $t^i A_{\leq i}$. Let $\bar{A} = A/pA$. As in [5, Section 3] we choose r so big that $x^{p^r} \in \text{gr}\,\bar{A}$ for all $x \in \text{hull}_\nabla(\text{gr}\,\bar{A})$. Put $\bar{C} = (\text{gr}\,\bar{A})^{\bar{G}_r}$. By [2, Thm. 30] the algebra $\mathcal{A}/t\mathcal{A} = \text{gr}\,A$ is finitely generated, so $\mathcal{A}$ is finitely generated. By [2, Thm. 35] the map $\text{gr}\,A \to \text{gr}\,\bar{A}$ is power surjective. Then so is the map $\mathcal{A} \to \text{gr}\,\bar{A}$, because $\mathcal{A} \to \text{gr}\,A$ is surjective. Now take a finitely

generated G invariant subalgebra $\mathcal{C}$ of the inverse image of $\bar{C}$ in $\mathcal{A}$ in such a way that $\mathcal{C} \to \bar{C}$ is power surjective. By theorem 2.11 the map $H^{\text{even}}(G, \mathcal{C}) \to H^0(G, H^*(\bar{G}_r, \bar{C}))$ is noetherian. By [1, Theorem 1.5, Remark 1.5.1] the $H^*(\bar{G}_r, \bar{C})$-module $H^*(\bar{G}_r, \operatorname{gr} \bar{A})$ is noetherian and by [2, Theorems 9, 12] it follows that $H^0(G, H^*(\bar{G}_r, \bar{C})) \to H^0(G, H^*(\bar{G}_r, \operatorname{gr} \bar{A}))$ is noetherian. Then so is $H^{\text{even}}(G, \mathcal{C}) \to H^0(G, H^*(\bar{G}_r, \operatorname{gr} \bar{A}))$, hence also $H^{\text{even}}(G, \mathcal{A}) \to H^0(G, H^*(\bar{G}_r, \operatorname{gr} \bar{A}))$. This is what is needed to argue as in [5, 4.10] that $H^{\text{even}}(G, \mathcal{A}) \to H^*(G, \operatorname{gr} \bar{A})$ is noetherian. And then one concludes as in [5] that $H^{\text{even}}(G, \mathcal{A}) \to H^*(G, \bar{A})$ is noetherian. But $\mathcal{A} \to \bar{A}$ factors through A. It follows that $H^{\text{even}}(G, A) \to H^*(G, \bar{A})$ is noetherian. As p was an arbitrary prime, [2, Thm. 49], or rather the last item of [7, Theorem 10.5], applies. $\qquad\square$

References

[1] Eric Friedlander and Andrei Suslin, *Cohomology of finite group scheme over a field*, Invent. Math 127 (1997) 209–270.

[2] Vincent Franjou and Wilberd van der Kallen, *Power reductivity over an arbitrary base*, Documenta Mathematica, Extra Volume Suslin (2010), pp. 171-195.

[3] Jens Carsten Jantzen, *Representations of algebraic groups.* Second edition. Mathematical Surveys and Monographs, 107. American Mathematical Society, Providence, RI, 2003.

[4] A. Touzé, *Universal classes for algebraic groups*, Duke Mathematical Journal 151 (2010), 219–249.

[5] Wilberd van der Kallen, *Cohomology with Grosshans graded coefficients*, In: Invariant Theory in All Characteristics, Edited by: H. E. A. Eddy Campbell and David L. Wehlau, CRM Proceedings and Lecture Notes, Volume 35 (2004) 127-138, Amer. Math. Soc., Providence, RI, 2004.

[SGA3] M. Demazure, A. Grothendieck, *Schémas en groupes* I, II, III, Lecture Notes in Math **151, 152, 153**, Springer-Verlag, New York (1970) and new edition in Documents Mathématiques **7, 8** of Société Mathématique de France (2011).

[6] Antoine Touzé and Wilberd van der Kallen, *Bifunctor cohomology and Cohomological finite generation for reductive groups* , Duke Mathematical Journal 151 (2010), 251–278.

[7] Wilberd van der Kallen, *Good Grosshans filtration in a family*, arXiv:1109.5822v3